Ej 1

$$\tilde{A} = \left[\begin{array}{ccc|c} -1.2 & 3.3 & 0.3 & -4.2 \\ 24 & -15 & 7 & 46 \\ 2.4 & -0.16 & 0.74 & 3.3 \end{array}\right]$$

$\{Gr(10,2,*), rd\}$

$$m_{21} = \frac{24}{-1,2} = -20$$

$$m_{31} = \frac{2,4}{-1,2} = -2$$

$$a_{22}^{(2)} = a_{22}^{(1)} - m_{21} * a_{12}^{(1)} = -15 - \underbrace{(-20)*3,3}_{66} = 51$$

$$a_{23}^{(2)} = a_{23}^{(1)} - m_{21} * a_{13}^{(1)} = 7 - \underbrace{(-20)*0,3}_{6} = 13$$

$$a_{24}^{(2)} = a_{24}^{(1)} - m_{21} * a_{14}^{(1)} = +46 - \underbrace{(-20)*(-4,2)}_{84} = \cancel{130}\ -38$$

$$a_{32}^{(2)} = a_{32}^{(1)} - m_{31} * a_{12}^{(1)} = -0,16 - \underbrace{(-2)*3,3}_{6,6} = 6.44 \underset{6}{=} 6,4$$

$$a_{33}^{(2)} = a_{33}^{(1)} - m_{31} * a_{13}^{(1)} = 0.74 - \underbrace{(-2)*0.3}_{0,6} = 1.34 \underset{6}{=} 1,3$$

$$a_{34}^{(2)} = a_{34}^{(1)} - m_{31} * a_{14}^{(1)} = 3,3 - \underbrace{(-2)*(-4,2)}_{8,4} = -5,1$$

$$\left[\begin{array}{ccc|c} -1.2 & 3.3 & 0.3 & -4.2 \\ (-20) & 51 & 13 & -38 \\ (-2) & 6.4 & 1.3 & -5.1 \end{array}\right] = \tilde{A}^{(2)}$$

$$m_{32} = \frac{6.4}{51} = 0.1255 \underset{6}{=} 0.13$$

$$a_{33}^{(3)} = a_{33}^{(2)} - m_{32} * a_{23}^{(2)} = 1.3 - \underbrace{0.13*13}_{1.69} \underset{6}{=} 1.3 - 1.7 \underset{6}{=} -0.4$$

$$a_{34}^{(3)} = a_{34}^{(2)} - m_{32} * a_{24}^{(2)} = -5.1 - \underbrace{0.13 * \overset{-38}{\cancel{130}}}_{4,9} \underset{6}{=} -5.1 + 4.9 \underset{6}{=} -0.2$$

$$\left[\begin{array}{ccc|c} -1.2 & 3.3 & 0.3 & -4,2 \\ (-20) & 51 & 13 & -38 \\ (-2) & (0.13) & -0.4 & -0.2 \end{array}\right]$$

Ahora sustitución inversa $x_3 = \dfrac{a_{34}^{(3)}}{a_{33}^{(3)}} = \dfrac{-0.2}{-0.4} = 0.5$

$$x_2 = (a_{24}^{(2)} - a_{23}^{(2)} * x_3)/a_{22}^{(2)} = (-38 - \underbrace{13*0.5}_{6.5})/51 = -44.5/51 = -45/51 = 0.88$$

$$x_1 = (a_{14}^{(1)} - a_{13}^{(1)} * x_3 - a_{12}^{(1)} * x_2)/a_{11}^{(1)} = (-4.2 - 0.3*0.5 - 3.3*(-0.88))/(-1.2)$$

$$x_1 = (-4.2 - 0.15 + 2.9)/(-1.2) \Rightarrow (-4.4 + 2.9)/(-1.2) = \frac{-1.5}{-1.2} = 1.3$$

$$\Rightarrow (-4.2 + 2.8)/(-1.2) = \frac{-1.4}{-1.2} = 1.2$$

Ambas son válidas

$$\tilde{x} = (1.3 ; -0.88 ; 0.5) \quad L = \begin{pmatrix} 1 & 0 & 0 \\ -20 & 1 & 0 \\ -2 & 0.13 & 1 \end{pmatrix} \quad U = \begin{pmatrix} -1.2 & 3.3 & 0.3 \\ 0 & 51 & 13 \\ 0 & 0 & -0.4 \end{pmatrix}$$

b) Ahora refinamiento iterativo (el residuo con doble precisión)

$$r = b - A\tilde{x} = \begin{pmatrix} -4,2 \\ 46 \\ 3.3 \end{pmatrix} - \begin{pmatrix} -1.2 & 3.3 & 0.3 \\ 24 & -15 & 7 \\ 2.4 & -0.16 & 0.74 \end{pmatrix} \begin{pmatrix} 1.3 \\ -0.88 \\ 0.5 \end{pmatrix} = \begin{pmatrix} -4.2 - (-4.314) \\ 46 - 47.9 \\ 3.3 - 3.631 \end{pmatrix} = \begin{pmatrix} 0.114 \\ -1.9 \\ -0.331 \end{pmatrix} \underset{6}{=} \begin{pmatrix} 0.11 \\ -1.9 \\ -0.3 \end{pmatrix}$$

de doble a simple precisión

$$Ly = r \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -20 & 1 & 0 \\ -2 & 0.13 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.11 \\ -1.9 \\ -0.33 \end{pmatrix} \Rightarrow$$

$y_1 = 0,11$

$y_2 = -1.9 + \underbrace{20*0.11}_{2.2} = 0.3$

$y_3 = -0.33 + \underbrace{2*0.11}_{0.22} - \underbrace{0.13*0.3}_{0.039}$

$y_3 = -0.11 - 0.039 = -0.149 \underset{6}{=} -0.15$

$$U\delta x = y \Rightarrow \begin{pmatrix} -1.2 & 3.3 & 0.3 \\ 0 & 51 & 13 \\ 0 & 0 & -0.4 \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix} = \begin{pmatrix} 0.11 \\ 0.3 \\ -0.15 \end{pmatrix}$$

$$\delta x_3 = \frac{-0.15}{-0.4} = 0.38 \quad ; \quad \delta x_2 = (0.3 - \underbrace{13*0.38}_{4.9})/51 = \frac{-4.6}{51} = -0.09$$

$$\delta x_1 = (0.11 - \underbrace{0.3*0.38}_{0.11} - \underbrace{3.3*(-0.09)}_{0.3})/(-1.2) = -0.25 \Rightarrow \delta x = (-0.25 ; -0.09 ; 0.38)$$

$$\tilde{\tilde{x}} = \tilde{x} + \delta x = (1.1 ; -0.97 ; 0.88) \rightarrow \text{es una mejora}$$

estimación $K(A) \approx \dfrac{\|\delta x\|}{\|\tilde{x}\|} 10^2 = \dfrac{0.38}{1.3} \times 10^2 = 29$

mala estimación pq $\dfrac{\|\delta x\|}{\|\tilde{x}\|} = 0.29 > 0.05$

Ej 2 - Cero de $f(x) = 8x^3 - e^x$.

a) $g(x) = \frac{e^{x/3}}{2}$; $x^* = \frac{e^{x^*/3}}{2} \Rightarrow 2x^* = e^{x^*/3} \Rightarrow (2x^*)^3 = e^{x^*} \Rightarrow 8x^{*3} = e^{x^*} \Rightarrow \underbrace{8x^{*3} - e^{x^*} = 0}_{x^* \text{ es cero de la } f(x)}$

Cálculo: $x_0 = 1$ (uso 6 dígitos decimales) $x_1 = 0.687806$

$x_2 = 0.630940$; $x_3 = 0.617032$; $x_4 = 0.614178$; $x_5 = 0.613594$

$x_6 = 0.613475$; $x_7 = 0.613451$; $x_8 = 0.613446$; $|x_6 - x_7| = 2.4 \times 10^{-5}$

Posible intervalo: $[a;b] = [0.5;1]$ (la sucesión se encuentra confinada en dicho intervalo).

Veamos si satisface las condiciones del TPF.

Primero g es continua en $[0.5;1]$, es más lo es en todo $\mathbb{R}$. ✓

Segundo g es monótona creciente porque lo es e^x. $g(0.5) = 0.58068 \in [0.5;1]$

$g(1) = 0.687806 \in [0.5;1]$

$\Rightarrow g: [0.5;1] \to [0.58;0.70] \subset [0.5;1]$ ✓

$g'(x) = \frac{1}{3} \cdot \frac{1}{2} e^{x/3} = \frac{e^{x/3}}{6}$; $g'(1) = 0.2326$; $g'(0.5) = 0.19689$

$L = \max\limits_{0.5<x<1} \{|g'(x)|\} = 0.24 < 1$ Tercera condición

Se satisfacen las 3 condiciones

Además $|x_7 - x^*| \leq \frac{L}{(1-L)} |x_7 - x_8| \leq \frac{L}{(1-L)} |x_7 - x_8| \leq \frac{0.24}{0.76} \cdot 2.4 \times 10^{-5} < \underline{10^{-4}}$

$x^* = 0.613444$. El orden de convergencia es uno pq $g'(x^*) = 0.2045 \neq 0$

y la constante asintótica es $CA = \frac{g'(x^*)}{1!} = 0.2045$.

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b) Por Newton-Raphson $\Rightarrow g_{NR}(x) = x - \frac{f(x)}{f'(x)} = x - \frac{(8x^3 - e^x)}{(24x^2 - e^x)}$

Cálculo: $x_0 = 1$; $x_1 = 0.751819$; $x_2 = 0.640085$; $x_3 = 0.614715$; $x_4 = 0.613447$

$x_5 = 0.613444$ (corto aquí) Posible intervalo: $[a;b] = [0.5;1]$ (el mismo)

g_{NR} posee un extremo local (mínimo) en x^* ($g'_{NR}(x^*) = \frac{f f''}{f'^2} = 0$)

$g_{NR}(1) = 0.751819 \in [0.5;1]$

$g_{NR}(0.5) = 0.649088 \in [0.5;1]$

$x^* \in [0.5;1]$

g_{NR} es continua en $[0.5;1]$

$g'(x)$ es continua y monótona en $[0.5;1]$ $g'(0.5) = -0.76582$

$g'(1) = 0.528062$

$g' = \frac{f f''}{f'^2} = \frac{(8x^3 - e^x)(48x - e^x)}{(24x^2 - e^x)}$

$L = \max\limits_{0.5<x<1} \{|g'(x)|\} = 0.77 < 1$

El orden de convergencia es 2 ($g'(x^*) = 0$) y la cte asintótica es $\frac{g''(x^*)}{2!}$

$g'' = (g')' = \left(\frac{f f''}{f'^2}\right)' = \frac{(f f'')' f'^2 - f f'' (f'^2)'}{f'^4} = \frac{f' f'' f'^2 + f f''' f'^2 - 2 f' f f'' f''^2}{f'^4}$

$g''(x^*) = \frac{f'(x^*) f''(x^*) f'^2(x^*)}{f'^4(x^*)}$ los términos con $f(x^*) = 0$ $\Rightarrow g''(x^*) = \frac{f''(x^*)}{f'(x^*)} = \frac{(48x^* - e^{x^*})}{(24x^{*2} - e^{x^*})} = 3{,}84$

$\Rightarrow CA = 1{,}92$

Ej 3 a) Sea $f(x)$ tq

x	$f(x)$	$f'(x)$
0	4	0
1	5	4

Hallar el $P_{(x)}$ de menor grado. Hay $N=4$ condiciones $\Rightarrow P \in \mathcal{P}_3$ (o sea es de grado 3)

Lo calculamos por diferencia divididas

$$\begin{array}{ll} 0 & 4 \\ 0 & 4 \quad 0 = f'(0) \\ 1 & 5 \quad 1 = \frac{f(1)-f(0)}{1-0} \quad \frac{1-0}{1-0}=1 \\ 1 & 5 \quad 4 = f'(1) \quad \frac{4-1}{1-0}=3 \quad \frac{3-1}{1-0}=2 \end{array}$$

$P(x) = 4 + 0(x-0) + 1(x-0)^2 + 2(x-0)^2(x-1)$

$P(x) = 4 + x^2 + 2x^2(x-1)$

$P(x) = 2x^3 - x^2 + 4$

Verifico

$P(0) = 2\cdot 0^3 - 0 + 4 = 4$ ✓

$P(1) = 2\cdot 1^3 - 1^2 + 4 = 5$ ✓

$P'(x) = 6x^2 - 2x$

$P'(0) = 6\cdot 0^2 - 2\cdot 0 = 0$ ✓

$P'(1) = 6\cdot 1^2 - 2\cdot 1 = 4$ ✓

Calculamos $P(0.5)$

$P(0.5) = 2\cdot 0.5^3 - 0.5^2 + 4 = 4$

b) $f(x) = x^4 + 4$; $f(0.5) = 4{,}0625$

Error verd $= |f(0{,}5) - P(0{,}5)| = 0{,}0625$

Cota del error $= |f(x) - P(x)| \leq \left|\frac{f^{(4)}(\xi)}{4!}\right| |x-0|^2 |x-1|^2$

siendo $f^{(4)}(x) = 24$

$f'(x) = 4x^3$; $f'' = 12x^2$

$f'''(x) = 24x$

$4! = 24$

Para $x = 0.5$ $\Rightarrow$ cota $= |f(0.5) - P(0.5)| \leq \frac{24}{24} \times 0.5^2 \times 0.5^2 = 0{,}0625$

En este caso el error verdadero y la cota del error en $x=0{,}5$ coinciden. Se puede demostrar que lo mismo sucederá para todos los x de $[0;1]$