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Chapter 1

Transport phenomena

1.1 Mathematical introduction

An important relation is: if X is a quantity of a volume element which travels from position $\vec{r}$ to $\vec{r}+d\vec{r}$ in a time dt, the total differential dX is then given by:

$$dX = \frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy + \frac{\partial X}{\partial z}dz + \frac{\partial X}{\partial t}dt \;\Rightarrow\; \frac{dX}{dt} = \frac{\partial X}{\partial x}v_x + \frac{\partial X}{\partial y}v_y + \frac{\partial X}{\partial z}v_z + \frac{\partial X}{\partial t}$$

This results in general to: $\boxed{\frac{dX}{dt} = \frac{\partial X}{\partial t} + (\vec{v}\cdot\nabla)X}$.

From this follows that also holds: $\boxed{\frac{d}{dt}\iiint X d^3V = \frac{\partial}{\partial t}\iiint X d^3V + \oiint X(\vec{v}\cdot\vec{n})d^2A}$

where the volume V is surrounded by surface A. Some properties of the ∇ operator are:

$\text{div}(\phi\vec{v}) = \phi\text{div}\vec{v} + \text{grad}\phi\cdot\vec{v}$	$\text{rot}(\phi\vec{v}) = \phi\text{rot}\vec{v} + (\text{grad}\phi)\times\vec{v}$	$\text{rot grad}\phi = \vec{0}$
$\text{div}(\vec{u}\times\vec{v}) = \vec{v}\cdot(\text{rot}\vec{u}) - \vec{u}\cdot(\text{rot}\vec{v})$	$\text{rot rot}\vec{v} = \text{grad div}\vec{v} - \nabla^2\vec{v}$	$\text{div rot}\vec{v} = 0$
$\text{div grad}\phi = \nabla^2\phi$	$\nabla^2\vec{v} \equiv (\nabla^2 v_1, \nabla^2 v_2, \nabla^2 v_3)$	

Here, $\vec{v}$ is an arbitrary vector field and ϕ an arbitrary scalar field. Some important integral theorems are:

Gauss: $$\oiint(\vec{v}\cdot\vec{n})d^2A = \iiint(\text{div}\vec{v})d^3V$$

Stokes for a scalar field: $$\oint(\phi\cdot\vec{e}_{\rm t})ds = \iint(\vec{n}\times\text{grad}\phi)d^2A$$

Stokes for a vector field: $$\oint(\vec{v}\cdot\vec{e}_{\rm t})ds = \iint(\text{rot}\vec{v}\cdot\vec{n})d^2A$$

This results in: $$\oiint(\text{rot}\vec{v}\cdot\vec{n})d^2A = 0$$

Ostrogradsky: $$\oiint(\vec{n}\times\vec{v})d^2A = \iiint(\text{rot}\vec{v})d^3A$$

$$\oiint(\phi\vec{n})d^2A = \iiint(\text{grad}\phi)d^3V$$

Here, the orientable surface $\iint d^2A$ is limited by the Jordan curve $\oint ds$.

1.2 Conservation laws

On a volume work two types of forces:

1. The force $\vec{f}_0$ on each volume element. For gravity holds: $\vec{f}_0 = \varrho\vec{g}$.
2. Surface forces working only on the margins: $\vec{t}$. For these holds: $\vec{t} = \vec{n}\,\mathsf{T}$, where T is the *stress tensor*.

T can be split in a part $p\mathsf{I}$ representing the normal tensions and a part T' representing the shear stresses: $\mathsf{T} = \mathsf{T}' + p\mathsf{I}$, where I is the unit tensor. When viscous aspects can be ignored holds: div$\mathsf{T}$= $-$grad$p$.

When the flow velocity is $\vec{v}$ at position $\vec{r}$ holds on position $\vec{r} + d\vec{r}$:

$$\vec{v}(d\vec{r}\,) = \underbrace{\vec{v}(\vec{r}\,)}_{\text{translation}} + \underbrace{d\vec{r} \cdot (\text{grad}\vec{v}\,)}_{\text{rotation, deformation, dilatation}}$$

The quantity L:=grad$\vec{v}$ can be split in a symmetric part D and an antisymmetric part W. $\mathsf{L} = \mathsf{D} + \mathsf{W}$ with

$$D_{ij} := \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \ , \ W_{ij} := \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

When the rotation or *vorticity* $\vec{\omega} = \text{rot}\vec{v}$ is introduced holds: $W_{ij} = \frac{1}{2}\varepsilon_{ijk}\omega_k$. $\vec{\omega}$ represents the local rotation velocity: $\vec{dr} \cdot \mathsf{W} = \frac{1}{2}\omega \times \vec{dr}$.

For a *Newtonian liquid* holds: $\mathsf{T}' = 2\eta\mathsf{D}$. Here, η is the dynamical viscosity. This is related to the shear stress τ by:

$$\tau_{ij} = \eta \frac{\partial v_i}{\partial x_j}$$

For compressible media can be stated: $\mathsf{T}' = (\eta' \text{div}\vec{v}\,)\mathsf{I} + 2\eta\mathsf{D}$. From equating the thermodynamical and mechanical pressure it follows: $3\eta' + 2\eta = 0$. If the viscosity is constant holds: $\text{div}(2\mathsf{D}) = \nabla^2\vec{v} + \text{grad div}\vec{v}$.

The conservation laws for mass, momentum and energy for continuous media can be written in both integral and differential form. They are:

Integral notation:

1. Conservation of mass: $\displaystyle\frac{\partial}{\partial t}\iiint \varrho d^3V + \oiint \varrho(\vec{v} \cdot \vec{n}\,)d^2A = 0$
2. Conservation of momentum: $\displaystyle\frac{\partial}{\partial t}\iiint \varrho\vec{v} d^3V + \oiint \varrho\vec{v}(\vec{v} \cdot \vec{n}\,)d^2A = \iiint f_0 d^3V + \oiint \vec{n} \cdot T d^2A$
3. Conservation of torque: $\displaystyle\frac{\partial}{\partial t}\iiint \varrho(\vec{r} \times \vec{v})d^3V + \oiint \varrho(\vec{r} \times \vec{v})(\vec{v} \cdot \vec{n}\,)d^2A =$
$$\iiint \vec{r} \times \vec{f_0} d^3V + \oiint \vec{r} \times \vec{n} \cdot T d^2A$$
4. Conservation of energy: $\displaystyle\frac{\partial}{\partial t}\iiint (\tfrac{1}{2}v^2 + e)\varrho d^3V + \oiint (\tfrac{1}{2}v^2 + e)\varrho(\vec{v} \cdot \vec{n}\,)d^2A =$
$$-\oiint (\vec{q} \cdot \vec{n}\,)d^2A + \iiint (\vec{v} \cdot \vec{f_0})d^3V + \oiint (\vec{v} \cdot \vec{n}\ \mathsf{T})d^2A$$

Differential notation:

1. Conservation of mass: $\displaystyle\frac{\partial\varrho}{\partial t} + \text{div} \cdot (\varrho\vec{v}\,) = 0$
2. Conservation of momentum: $\displaystyle\varrho\frac{\partial\vec{v}}{\partial t} + (\varrho\vec{v} \cdot \nabla)\vec{v} = \vec{f_0} + \text{div}\mathsf{T} = \vec{f_0} - \text{grad}p + \text{div}\mathsf{T}'$
3. Conservation of energy: $\displaystyle\varrho T\frac{ds}{dt} = \varrho\frac{de}{dt} - \frac{p}{\varrho}\frac{d\varrho}{dt} = -\text{div}\vec{q} + \mathsf{T}' : \mathsf{D}$

Here, e is the internal energy per unit of mass E/m and s is the entropy per unit of mass S/m. $\vec{q} = -\kappa\vec{\nabla}T$ is the heat flow. Further holds:

$$p = -\frac{\partial E}{\partial V} = -\frac{\partial e}{\partial 1/\varrho} \quad , \quad T = \frac{\partial E}{\partial S} = \frac{\partial e}{\partial s}$$

so

$$C_V = \left(\frac{\partial e}{\partial T}\right)_V \quad \text{and} \quad C_p = \left(\frac{\partial h}{\partial T}\right)_p$$

with $h = H/m$ the enthalpy per unit of mass.

From this one can derive the *Navier-Stokes* equations for an incompressible, viscous and heat-conducting medium:

$$\begin{aligned} \text{div}\vec{v} &= 0 \\ \varrho\frac{\partial \vec{v}}{\partial t} + \varrho(\vec{v}\cdot\nabla)\vec{v} &= \varrho\vec{g} - \text{grad}p + \eta\nabla^2\vec{v} \\ \varrho C\frac{\partial T}{\partial t} + \varrho C(\vec{v}\cdot\nabla)T &= \kappa\nabla^2 T + 2\eta\mathsf{D} : \mathsf{D} \end{aligned}$$

with C the thermal heat capacity. The force $\vec{F}$ on an object within a flow, when viscous effects are limited to the boundary layer, can be obtained using the momentum law. If a surface A surrounds the object outside the boundary layer holds:

$$\vec{F} = -\oiint [p\vec{n} + \varrho\vec{v}(\vec{v}\cdot\vec{n})]d^2A$$

Integral notation for a non inertial reference:

1. Conservation of mass: $\dfrac{\partial}{\partial t}\iiint \varrho d^3V + \oiint \varrho(\vec{w}\cdot\vec{n})d^2A = 0$
2. Conservation of momentum: $\dfrac{\partial}{\partial t}\iiint \varrho\vec{c}d^3V + \oiint \varrho\vec{c}(\vec{w}\cdot\vec{n})d^2A + \Omega\times\iiint \rho\vec{c}d^3V =$

$$\iiint f_0 d^3V + \oiint \vec{n}\cdot T d^2A$$

Where $\vec{c} = \vec{v} + \vec{w} + \vec{\Omega}\times\vec{r}$. $\vec{c}$: Absolute velocity; $\vec{v}$: Frame of reference velocity; $\vec{w}$: Relative velocity; $\vec{\Omega}$: Frame of reference angular velocity;

1.3 Bernoulli's equations

Starting with the momentum equation one can find for a non-viscous medium for stationary flows, with

$$(\vec{v}\cdot\text{grad})\vec{v} = \tfrac{1}{2}\text{grad}(v^2) + (\text{rot}\vec{v})\times\vec{v}$$

and the potential equation $\vec{g} = -\text{grad}(gh)$ that:

$$\tfrac{1}{2}v^2 + gh + \int\frac{dp}{\varrho} = \text{constant along a streamline}$$

For compressible flows holds: $\frac{1}{2}v^2 + gh + p/\varrho$ =constant along a line of flow. If also holds $\text{rot}\vec{v} = 0$ and the entropy is equal on each streamline holds $\frac{1}{2}v^2 + gh + \int dp/\varrho$ =constant everywhere. For incompressible flows this becomes: $\frac{1}{2}v^2 + gh + p/\varrho$ =constant everywhere. For ideal gases with constant C_p and C_V holds, with $\gamma = C_p/C_V$:

$$\tfrac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\frac{p}{\varrho} = \tfrac{1}{2}v^2 + \frac{c^2}{\gamma-1} = \text{constant}$$

With a velocity potential defined by $\vec{v} = \text{grad}\phi$ holds for instationary flows:

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}v^2 + gh + \int\frac{dp}{\varrho} = \text{constant everywhere}$$

1.4 Characterising of flows by dimensionless numbers

The advantage of dimensionless numbers is that they make model experiments possible: one has to make the dimensionless numbers which are important for the specific experiment equal for both model and the real situation. One can also deduce functional equalities without solving the differential equations. Some dimensionless numbers are given by:

Strouhal:	$\mathrm{Sr} = \dfrac{\omega L}{v}$	Froude:	$\mathrm{Fr} = \dfrac{v^2}{gL}$	Mach:	$\mathrm{Ma} = \dfrac{v}{c}$
Fourier:	$\mathrm{Fo} = \dfrac{a}{\omega L^2}$	Péclet:	$\mathrm{Pe} = \dfrac{vL}{a}$	Reynolds:	$\mathrm{Re} = \dfrac{vL}{\nu}$
Prandtl:	$\mathrm{Pr} = \dfrac{\nu}{a}$	Nusselt:	$\mathrm{Nu} = \dfrac{L\alpha}{\kappa}$	Eckert:	$\mathrm{Ec} = \dfrac{v^2}{c\Delta T}$

Here, $\nu = \eta/\varrho$ is the *kinematic viscosity*, c is the speed of sound and L is a characteristic length of the system. α follows from the equation for heat transport $\kappa\partial_y T = \alpha\Delta T$ and $a = \kappa/\varrho c$ is the thermal diffusion coefficient.

These numbers can be interpreted as follows:

- Re: (stationary inertial forces)/(viscous forces)
- Sr: (non-stationary inertial forces)/(stationary inertial forces)
- Fr: (stationary inertial forces)/(gravity)
- Fo: (heat conductance)/(non-stationary change in enthalpy)
- Pe: (convective heat transport)/(heat conductance)
- Ec: (viscous dissipation)/(convective heat transport)
- Ma: (velocity)/(speed of sound): objects moving faster than approximately Ma = 0,8 produce shockwaves which propagate with an angle θ with the velocity of the object. For this angle holds Ma$= 1/\arctan(\theta)$.
- Pr and Nu are related to specific materials.

Now, the dimensionless Navier-Stokes equation becomes, with $x' = x/L$, $\vec{v}\,' = \vec{v}/V$, $\mathrm{grad}' = L\mathrm{grad}$, $\nabla'^2 = L^2\nabla^2$ and $t' = t\omega$:

$$\mathrm{Sr}\frac{\partial\vec{v}\,'}{\partial t'} + (\vec{v}\,'\cdot\nabla')\vec{v}\,' = -\mathrm{grad}'p + \frac{\vec{g}}{\mathrm{Fr}} + \frac{\nabla'^2\vec{v}\,'}{\mathrm{Re}}$$

1.5 Tube flows

For tube flows holds: they are laminar if Re$<$ 2300 with dimension of length the diameter of the tube, and turbulent if Re is larger. For an incompressible laminar flow through a straight, circular tube holds for the velocity profile:

$$v(r) = -\frac{1}{4\eta}\frac{dp}{dx}(R^2 - r^2)$$

For the volume flow holds: $\Phi_V = \int\limits_0^R v(r)2\pi r dr = -\dfrac{\pi}{8\eta}\dfrac{dp}{dx}R^4$

The *entrance length* L_e is given by:

1. $500 < \mathrm{Re}_D < 2300$: $L_\mathrm{e}/2R = 0.056\mathrm{Re}_D$

2. Re > 2300: $L_{\rm e}/2R \approx 50$

For gas transport at low pressures (Knudsen-gas) holds: $\Phi_V = \dfrac{4R^3\alpha\sqrt{\pi}}{3}\dfrac{dp}{dx}$

For flows at a small Re holds: $\nabla p = \eta\nabla^2\vec{v}$ and div$\vec{v} = 0$. For the total force on a sphere with radius R in a flow then holds: $F = 6\pi\eta Rv$. For large Re holds for the force on a surface A: $F = \frac{1}{2}C_W A\varrho v^2$.

1.6 Potential theory

The *circulation* Γ is defined as: $\Gamma = \oint(\vec{v}\cdot\vec{e}_{\rm t})ds = \iint(\text{rot}\vec{v}\,)\cdot\vec{n}d^2A = \iint(\vec{\omega}\cdot\vec{n}\,)d^2A$

For non viscous media, if $p = p(\varrho)$ and all forces are conservative, Kelvin's theorem can be derived:

$$\frac{d\Gamma}{dt} = 0$$

For rotationless flows a velocity potential $\vec{v} = \text{grad}\phi$ can be introduced. In the incompressible case follows from conservation of mass $\nabla^2\phi = 0$. For a 2-dimensional flow a flow function $\psi(x,y)$ can be defined: with Φ_{AB} the amount of liquid flowing through a curve s between the points A and B:

$$\Phi_{AB} = \int\limits_A^B(\vec{v}\cdot\vec{n}\,)ds = \int\limits_A^B(v_x dy - v_y dx)$$

and the definitions $v_x = \partial\psi/\partial y$, $v_y = -\partial\psi/\partial x$ holds: $\Phi_{AB} = \psi(B) - \psi(A)$. In general holds:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega_z$$

In polar coordinates holds:

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r}\quad,\quad v_\theta = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

For source flows with power Q in $(x,y) = (0,0)$ holds: $\phi = \dfrac{Q}{2\pi}\ln(r)$ so that $v_r = Q/2\pi r$, $v_\theta = 0$.

For a dipole of strength Q in $x = a$ and strength $-Q$ in $x = -a$ follows from superposition: $\phi = -Qax/2\pi r^2$ where Qa is the dipole strength. For a vortex holds: $\phi = \Gamma\theta/2\pi$.

If an object is surrounded by an uniform main flow with $\vec{v} = v\vec{e}_x$ and such a large Re that viscous effects are limited to the boundary layer holds: $F_x = 0$ and $F_y = -\varrho\Gamma v$. The statement that $F_x = 0$ is d'Alembert's paradox and originates from the neglection of viscous effects. The lift F_y is also created by η because $\Gamma \neq 0$ due to viscous effects. Henxe rotating bodies also create a force perpendicular to their direction of motion: the *Magnus effect*.

1.7 Boundary layers

1.7.1 Flow boundary layers

If for the thickness of the boundary layer holds: $\delta \ll L$ holds: $\delta \approx L/\sqrt{\text{Re}}$. With v_∞ the velocity of the main flow it follows for the velocity $v_y \perp$ the surface: $v_y L \approx \delta v_\infty$. Blasius' equation for the boundary layer is, with $v_y/v_\infty = f(y/\delta)$: $2f''' + ff'' = 0$ with boundary conditions $f(0) = f'(0) = 0$, $f'(\infty) = 1$. From this follows: $C_W = 0.664\ \text{Re}_x^{-1/2}$.

The momentum theorem of Von Karman for the boundary layer is: $\dfrac{d}{dx}(\vartheta v^2) + \delta^* v\dfrac{dv}{dx} = \dfrac{\tau_0}{\varrho}$

where the displacement thickness $\delta^* v$ and the momentum thickness ϑv^2 are given by:

$$\vartheta v^2 = \int\limits_0^\infty (v - v_x) v_x dy \quad , \quad \delta^* v = \int\limits_0^\infty (v - v_x) dy \ \ \text{and} \ \ \tau_0 = -\eta \left. \frac{\partial v_x}{\partial y} \right|_{y=0}$$

The boundary layer is released from the surface if $\left(\frac{\partial v_x}{\partial y} \right)_{y=0} = 0$. This is equivalent with $\frac{dp}{dx} = \frac{12\eta v_\infty}{\delta^2}$.

1.7.2 Temperature boundary layers

If the thickness of the temperature boundary layer $\delta_T \ll L$ holds:

1. If $\mathrm{Pr} \leq 1$: $\delta/\delta_T \approx \sqrt{\mathrm{Pr}}$.
2. If $\mathrm{Pr} \gg 1$: $\delta/\delta_T \approx \sqrt[3]{\mathrm{Pr}}$.

1.8 Heat conductance

For non-stationairy heat conductance in one dimension without flow holds:

$$\frac{\partial T}{\partial t} = \frac{\kappa}{\varrho c} \frac{\partial^2 T}{\partial x^2} + \Phi$$

where Φ is a source term. If $\Phi = 0$ the solutions for harmonic oscillations at $x = 0$ are:

$$\frac{T - T_\infty}{T_{\max} - T_\infty} = \exp\left(-\frac{x}{D}\right) \cos\left(\omega t - \frac{x}{D}\right)$$

with $D = \sqrt{2\kappa/\omega\varrho c}$. At $x = \pi D$ the temperature variation is in anti-phase with the surface. The one-dimensional solution at $\Phi = 0$ is

$$T(x,t) = \frac{1}{2\sqrt{\pi a t}} \exp\left(-\frac{x^2}{4at}\right)$$

This is mathematical equivalent to the diffusion problem:

$$\frac{\partial n}{\partial t} = D\nabla^2 n + P - A$$

where P is the production of and A the discharge of particles. The flow density $J = -D\nabla n$.

1.9 Turbulence

The time scale of turbulent velocity variations τ_t is of the order of: $\tau_\mathrm{t} = \tau\sqrt{\mathrm{Re}}/\mathrm{Ma}^2$ with τ the molecular time scale. For the velocity of the particles holds: $v(t) = \langle v \rangle + v'(t)$ with $\langle v'(t) \rangle = 0$. The Navier-Stokes equation now becomes:

$$\frac{\partial \langle \vec{v} \rangle}{\partial t} + (\langle \vec{v} \rangle \cdot \nabla) \langle \vec{v} \rangle = -\frac{\nabla \langle p \rangle}{\varrho} + \nu \nabla^2 \langle \vec{v} \rangle + \frac{\mathrm{div} \mathsf{S}_R}{\varrho}$$

where $\mathsf{S}_{Rij} = -\varrho \langle v_i v_j \rangle$ is the turbulent stress tensor. Boussinesq's assumption is: $\tau_{ij} = -\varrho \left\langle v_i' v_j' \right\rangle$. It is stated that, analogous to Newtonian media: $\mathsf{S}_R = 2\varrho\nu_t \langle \mathsf{D} \rangle$. Near a boundary holds: $\nu_t = 0$, far away of a boundary holds: $\nu_t \approx \nu \mathrm{Re}$.

1.10 Self organization

For a (semi) two-dimensional flow holds: $\dfrac{d\omega}{dt} = \dfrac{\partial\omega}{\partial t} + J(\omega,\psi) = \nu\nabla^2\omega$

With $J(\omega,\psi)$ the Jacobian. So if $\nu = 0$, ω is conserved. Further, the kinetic energy/mA and the enstrofy V are conserved: with $\vec{v} = \nabla \times (\vec{k}\psi)$

$$E \sim (\nabla\psi)^2 \sim \int\limits_0^\infty \mathcal{E}(k,t)dk = \text{constant} \quad , \quad V \sim (\nabla^2\psi)^2 \sim \int\limits_0^\infty k^2\mathcal{E}(k,t)dk = \text{constant}$$

From this follows that in a two-dimensional flow the energy flux goes towards large values of k: larger structures become larger at the expanse of smaller ones. In three-dimensional flows the situation is just the opposite.

The ∇-operator

In cartesian coordinates (x, y, z) holds:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \quad , \quad \mathrm{grad} f = \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\mathrm{div}\ \vec{a} = \vec{\nabla} \cdot \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \quad , \quad \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\mathrm{rot}\ \vec{a} = \vec{\nabla} \times \vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{e}_z$$

In cylinder coordinates (r, φ, z) holds:

$$\vec{\nabla} = \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial \varphi}\vec{e}_\varphi + \frac{\partial}{\partial z}\vec{e}_z \quad , \quad \mathrm{grad} f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\mathrm{div}\ \vec{a} = \frac{\partial a_r}{\partial r} + \frac{a_r}{r} + \frac{1}{r}\frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z} \quad , \quad \nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\mathrm{rot}\ \vec{a} = \left(\frac{1}{r}\frac{\partial a_z}{\partial \varphi} - \frac{\partial a_\varphi}{\partial z}\right)\vec{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\vec{e}_\varphi + \left(\frac{\partial a_\varphi}{\partial r} + \frac{a_\varphi}{r} - \frac{1}{r}\frac{\partial a_r}{\partial \varphi}\right)\vec{e}_z$$

In spherical coordinates (r, θ, φ) holds:

$$\begin{aligned}
\vec{\nabla} &= \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\vec{e}_\varphi \\
\mathrm{grad} f &= \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi \\
\mathrm{div}\ \vec{a} &= \frac{\partial a_r}{\partial r} + \frac{2a_r}{r} + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{a_\theta}{r\tan\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial \varphi} \\
\mathrm{rot}\ \vec{a} &= \left(\frac{1}{r}\frac{\partial a_\varphi}{\partial \theta} + \frac{a_\theta}{r\tan\theta} - \frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial \varphi}\right)\vec{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial \varphi} - \frac{\partial a_\varphi}{\partial r} - \frac{a_\varphi}{r}\right)\vec{e}_\theta + \\
&\quad \left(\frac{\partial a_\theta}{\partial r} + \frac{a_\theta}{r} - \frac{1}{r}\frac{\partial a_r}{\partial \theta}\right)\vec{e}_\varphi \\
\nabla^2 f &= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\tan\theta}\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}
\end{aligned}$$

General orthonormal curvelinear coordinates (u, v, w) can be obtained from cartesian coordinates by the transformation $\vec{x} = \vec{x}(u, v, w)$. The unit vectors are then given by:

$$\vec{e}_u = \frac{1}{h_1}\frac{\partial \vec{x}}{\partial u} \ , \quad \vec{e}_v = \frac{1}{h_2}\frac{\partial \vec{x}}{\partial v} \ , \quad \vec{e}_w = \frac{1}{h_3}\frac{\partial \vec{x}}{\partial w}$$

where the factors h_i set the norm to 1. Then holds:

$$\begin{aligned}
\mathrm{grad} f &= \frac{1}{h_1}\frac{\partial f}{\partial u}\vec{e}_u + \frac{1}{h_2}\frac{\partial f}{\partial v}\vec{e}_v + \frac{1}{h_3}\frac{\partial f}{\partial w}\vec{e}_w \\
\mathrm{div}\ \vec{a} &= \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u}(h_2h_3a_u) + \frac{\partial}{\partial v}(h_3h_1a_v) + \frac{\partial}{\partial w}(h_1h_2a_w)\right) \\
\mathrm{rot}\ \vec{a} &= \frac{1}{h_2h_3}\left(\frac{\partial(h_3a_w)}{\partial v} - \frac{\partial(h_2a_v)}{\partial w}\right)\vec{e}_u + \frac{1}{h_3h_1}\left(\frac{\partial(h_1a_u)}{\partial w} - \frac{\partial(h_3a_w)}{\partial u}\right)\vec{e}_v + \\
&\quad \frac{1}{h_1h_2}\left(\frac{\partial(h_2a_v)}{\partial u} - \frac{\partial(h_1a_u)}{\partial v}\right)\vec{e}_w \\
\nabla^2 f &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial w}\right)\right]
\end{aligned}$$

The SI units

Basic units

Quantity	Unit	Sym.
Length	metre	m
Mass	kilogram	kg
Time	second	s
Therm. temp.	kelvin	K
Electr. current	ampere	A
Luminous intens.	candela	cd
Amount of subst.	mol	mol

Extra units

Plane angle	radian	rad
solid angle	sterradian	sr

Derived units with special names

Quantity	Unit	Sym.	Derivation
Frequency	hertz	Hz	s^{-1}
Force	newton	N	$kg \cdot m \cdot s^{-2}$
Pressure	pascal	Pa	$N \cdot m^{-2}$
Energy	joule	J	$N \cdot m$
Power	watt	W	$J \cdot s^{-1}$
Charge	coulomb	C	$A \cdot s$
El. Potential	volt	V	$W \cdot A^{-1}$
El. Capacitance	farad	F	$C \cdot V^{-1}$
El. Resistance	ohm	Ω	$V \cdot A^{-1}$
El. Conductance	siemens	S	$A \cdot V^{-1}$
Mag. flux	weber	Wb	$V \cdot s$
Mag. flux density	tesla	T	$Wb \cdot m^{-2}$
Inductance	henry	H	$Wb \cdot A^{-1}$
Luminous flux	lumen	lm	$cd \cdot sr$
Illuminance	lux	lx	$lm \cdot m^{-2}$
Activity	bequerel	Bq	s^{-1}
Absorbed dose	gray	Gy	$J \cdot kg^{-1}$
Dose equivalent	sievert	Sv	$J \cdot kg^{-1}$

Prefixes

yotta	Y	10^{24}	giga	G	10^9	deci	d	10^{-1}	pico	p	10^{-12}
zetta	Z	10^{21}	mega	M	10^6	centi	c	10^{-2}	femto	f	10^{-15}
exa	E	10^{18}	kilo	k	10^3	milli	m	10^{-3}	atto	a	10^{-18}
peta	P	10^{15}	hecto	h	10^2	micro	μ	10^{-6}	zepto	z	10^{-21}
tera	T	10^{12}	deca	da	10	nano	n	10^{-9}	yocto	y	10^{-24}

Lo anterior es un extracto del formulario escrito por Johan Wevers (johanw@vulcan.xs4all.nl). El texto completo se halla en: ***http://www.xs4all.nl/ johanw/contents.html***