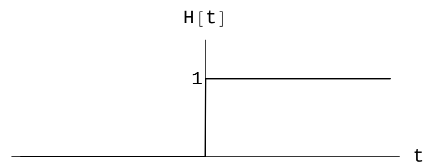
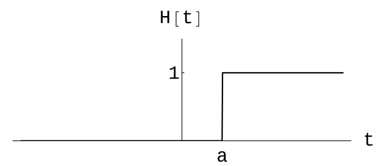
1. Transformada de Laplace

1.1. Función de Heaviside

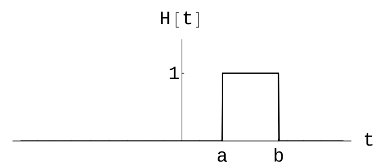
$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$H(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$$



$$H(t-a) - H(t-b) = \begin{cases} 0 & t < a \\ 1 & a \leq t \leq b \\ 0 & t > b \end{cases}$$



1.2. Definición Transformada de Laplace

$t \in \mathbb{R} \wedge f(t)/f(t) = 0 \qquad \forall t < 0$:

$$F(p) = \int_0^{\infty} f(t) e^{-pt} dt$$

Observaciones:

- $t \in \mathbb{R}$
- $p \in \mathbb{C}$
- $F(p) \in \mathbb{C}$

1.3. Convergencia de la Transformada de Laplace

1.3.1. Convergencia simple (CV)

$$\left.\begin{array}{l} F(p_0) \in \text{ CV} \\ \Re e(p) > \Re e(p_0) \end{array}\right\} \Longrightarrow F(p) \in \text{ CV}$$

$$\left.\begin{array}{l} F(p_1) \notin \text{ CV} \\ \Re e(p_1) > \Re e(p) \end{array}\right\} \Longrightarrow F(p) \notin \text{ CV}$$

$$\alpha \in \text{ abscisa de CV } \Longleftrightarrow \begin{cases} \forall p / \Re e(p) > \alpha : & F(p) \in \text{ CV} \\ \forall p / \Re e(p) < \alpha : & F(p) \notin \text{ CV} \end{cases}$$

1.3.2. Convergencia absoluta (CA)

$$\left.\begin{array}{l} F(p_0) \in \text{ CA} \\ \\ \Re e(p) > \Re e(p_0) \end{array}\right\} \Longrightarrow F(p) \in \text{ CA}$$

$$\left.\begin{array}{l} F(p_1) \notin \text{ CA} \\ \\ \Re e(p_1) > \Re e(p) \end{array}\right\} \Longrightarrow F(p) \notin \text{ CA}$$

$$\beta \in \text{ abscisa de CA } \Longleftrightarrow \left\{\begin{array}{ll} \forall p / \Re e(p) > \beta : & F(p) \in \text{ CA} \\ \\ \forall p / \Re e(p) < \beta : & F(p) \notin \text{ CA} \end{array}\right.$$

$$\alpha \leq \beta$$

1.3.3. Convergencia uniforme (CU)

$$\left.\begin{array}{l} f(t) \in \text{ CPOE} \\ \\ \Re e(p) > x_0 \end{array}\right\} \Longrightarrow F(p) \in \text{ CU} \qquad \forall\, x > x_1 > x_0 > 0$$

1.4. Holomorfia

$$\left.\begin{array}{l} f(t) \in \text{ CPOE} \\ \\ \Re e(p) > x_0 \end{array}\right\} \Longrightarrow F(p) \in \text{ H} \qquad \forall\, x > x_1 > x_0 > 0$$

1.5. Tabla de transformadas

Condiciones	$f(t)$	$F(p)=\mathcal{L}(f(t))$	Condiciones	$f(t)$	$F(p)=\mathcal{L}(f(t))$
$\Re e(p) \geq 0$	$H(t)$	$\frac{1}{p}$	$\Re e(p) > 0$ $a \in \mathbb{R}$	$\sin(at)$	$\frac{a}{p^2+a^2}$
$\Re e(p) > a$ $a \in \mathbb{R}$	e^{at}	$\frac{1}{p-a}$	$\Re e(p) > 0$ $a \in \mathbb{R}$	$\cos(at)$	$\frac{p}{p^2+a^2}$
$\Re e(p) > -a$ $a \in \mathbb{R}$	e^{-at}	$\frac{1}{p+a}$	$\Re e(p) > \|a\|$ $a \in \mathbb{R}$	$\sinh(at)$	$\frac{a}{p^2-a^2}$
$\Re e(p) > 0$ $a \in \mathbb{R}$	e^{iat}	$\frac{1}{p-ia}$	$\Re e(p) > \|a\|$ $a \in \mathbb{R}$	$\cosh(at)$	$\frac{p}{p^2-a^2}$
$\Re e(p) > 0$ $a \in \mathbb{R}$	e^{-iat}	$\frac{1}{p+ia}$	$\Re e(p) > 0$ $\alpha \in \mathbb{R}$	t^α	$\frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$
	$\delta(t)$	1	$\Re e(p) > 0$ $n \in \mathbb{N}$	t^n	$\frac{n!}{p^{n+1}}$

1.6. Propiedades

Condiciones	$f(t)$	$F(p) = \mathcal{L}(f(t))$	Descripción
$\begin{cases} f_1(t)H(t) \longrightarrow F_1(p) \\ f_2(t)H(t) \longrightarrow F_2(p) \end{cases}$	$(a * f_1(t) + b * f_2(t))H(t)$	$a * F_1(p) + b * F_2(p)$	Linealidad
$\begin{cases} \Re e(p) > 0 \\ a > 0 \end{cases}$	$f(t-a)H(t-a)$	$e^{-ap}F(p)$	Desplazamiento de la original
$\Re e(p) > 0$	$e^{at}f(t)H(t)$	$F(p-a)$	Desplazamiento de la transformada
$\begin{cases} f(0^+) \in \text{ OE} \\ \exists \lim_{t\to 0^+} f(t) = f(0^+) \end{cases}$	$f'(t)H(t)$	$pF(p) - f(0^+)$	Derivada de la original
$\begin{cases} f(0^+) \in \text{ OE} \\ \exists \lim_{t\to 0^+} f(t) = f(0^+) \end{cases}$	$f^{(n)}(t)H(t)$	$p^nF(p) - \sum_{i=1}^{n} p^{n-i}f^{(i-1)}(0+)$	Derivada de la original - generalización
$f(t) \in \text{ OE}$	$-tf(t)H(t)$	$F'(p)$	Derivada de la transformada
$f(t) \in \text{ OE}$	$(-t)^n f(t)H(t)$	$F^{(n)}(p)$	Derivada de la transformada - generalización
$f(t) \in \text{ OE}$	$\int_0^t f(\tau)d\tau$	$\frac{F(p)}{p}$	Integral de la original
$\begin{cases} f(t) \in \text{ OE} \\ \lim_{t\to 0} \frac{f(t)}{t} \in \text{ finito} \end{cases}$	$\frac{f(t)}{t}H(t)$	$\int_p^\infty F(q)dq$	Integral de la transformada
	$f(\lvert k\rvert t)H(t)$	$\frac{1}{\lvert k\rvert} F\left(\frac{p}{\lvert k\rvert}\right)$	Cambio de escala

1.6.1. Teorema de Borel

$$\left.\begin{array}{l} f(t)H(t) \longrightarrow F(p) \\ \\ g(t)H(t) \longrightarrow G(p) \end{array}\right\} \Longrightarrow f(t) * g(t) \longrightarrow F(p)G(p)$$

1.6.2. Transformada de funciones periódicas

$$\left.\begin{array}{l} f(t) = f(t-T) \\ \\ f_1(T) = \begin{cases} f(t) & t \in [0,T] \\ \\ 0 & t > T \end{cases} \end{array}\right\} \Longrightarrow F(p) = \frac{F_1(p)}{1 - e^{-pT}}$$

1.6.3. Teorema del valor inicial

$$\left.\begin{array}{l} f(t) \in \text{ OE} \\ \\ f'(t) \in \text{ OE} \\ \\ f(t)H(t) \longrightarrow F(p) \end{array}\right\} \Longrightarrow \lim_{p \to \infty} pF(p) = f(0^+)$$

1.6.4. Teorema del valor final

$$\left.\begin{array}{l} f(t) \in \text{ OE} \\ \\ f'(t) \in \text{ OE} \\ \\ f(t)H(t) \longrightarrow F(p) \end{array}\right\} \Longrightarrow \lim_{p \to 0} pF(p) = \lim_{t \to \infty} f(t)$$

1.6.5. Delta de Dirac

$$\begin{cases} \delta(t) = \begin{cases} \infty & t = 0 \\ \\ 0 & t \neq 0 \end{cases} \\ \\ \displaystyle\int_{-\infty}^{+\infty} \delta(t)\, dt = 1 \end{cases}$$

$$\delta(t-a) = \begin{cases} \infty & t = a \\ \\ 0 & t \neq a \end{cases}$$

$$\int_{-\infty}^{+\infty} f(t)\delta(t)dt = f(0)$$

1.7. Antitransformación

1.7.1. Desarrollo de Heaviside

$$F(p) = \frac{P(p)}{Q(p)}$$

a) $Q(p)$ tiene solamente una cantidad finita n de raices simples a_k.
b) grado $P(p) <$ grado $Q(p)$.

$$\Longrightarrow \frac{P(p)}{Q(p)} \longleftarrow f(t)H(t) = \sum_{k=1}^{n} \frac{P(a_k)}{Q'(a_k)} e^{a_k t} H(t)$$

1.7.2. Teorema de inversión o de Riemann - Mellin

$$\left.\begin{array}{l} F(p) \in H \quad \forall p/\Re e(p) > \alpha \\ \\ b > \alpha \qquad b \in \mathbb{R} \text{ constante} \end{array}\right\} \Longrightarrow f(t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F(p) e^{pt}\, dp$$

1.7.3. Formula de los residuos

$$\left.\begin{array}{l} F(p) \in H \quad \forall p/\Re e(p) > 0 \\ \\ b > \alpha \qquad b \in \mathbb{R} \text{ constante} \\ \\ \{p_i\} \text{ singularidades aisladas} \quad i \in [1, n] \\ \\ \{r_j\} \text{ puntos de ramificación} \quad j \in [1, m] \\ \\ \lim_{p \to \infty} F(p) = 0 \end{array}\right\} \Longrightarrow f^*(t)H(t) = \left[\sum_{i=1}^{n} \text{Res}\left\{F(p)e^{pt}; p_i\right\} - \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{\Gamma_j} F(p) e^{pt}\, dp \right] H(t)$$